Functions several variables $f: \mathbb{R}^N \to \mathbb{R}^M$

$\mathrm{Dom} f \equiv$ points where the function is well-defined

$\mathrm{Im} f \equiv$ points $y \in \mathbb{R}^M$ such that there exists $x \in \mathbb{R}^N$ with $f(x) = y$

Example: Problem 8 ii)

$$f(x,y) = \frac{1}{xy} \qquad f: \underset{\mathrm{Dom} f}{\underset{\uparrow}{\mathbb{R}^2}} \longrightarrow \underset{\mathrm{Im} f.}{\underset{\uparrow}{\mathbb{R}}}$$

$$\mathrm{Dom} f = \{ (x,y) \in \mathbb{R}^2 \, ; \, xy \neq 0 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \, ; \, x \neq 0 \text{ or } y \neq 0 \} \subset \mathbb{R}^2$$

$$\mathrm{Im} f = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$$

Type of functions

Scalar functions $f : \mathbb{R}^N \to \mathbb{R}$ the image is a number

Vector functions $F : \mathbb{R}^N \to \mathbb{R}^M$ the image is a vector

In particular, **vector fields** if $N = M$

$F : \mathbb{R}^N \to \mathbb{R}^N$

Examples:

Parametric equations of a line in $\mathbb{R}^3$

$$\left.\begin{aligned} x &= t \\ y &= 2+3t \\ z &= 1+t \end{aligned}\right\} \qquad F(x,y,z) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

- $C : \mathbb{R} \to \mathbb{R}^3$

  $t \to C(t) \equiv$ giving the direction and magnitude of flux for electric current in the heart.

$$F(x,y,z) = (xy, x^2y, \log x)$$

Scalar functions $\equiv$ magnitudes

Vector fields $\equiv$ Velocity, etc.

Definition - Level curves

Let $A \subset \mathbb{R}^n$ be a subset and $f: A \to \mathbb{R}$

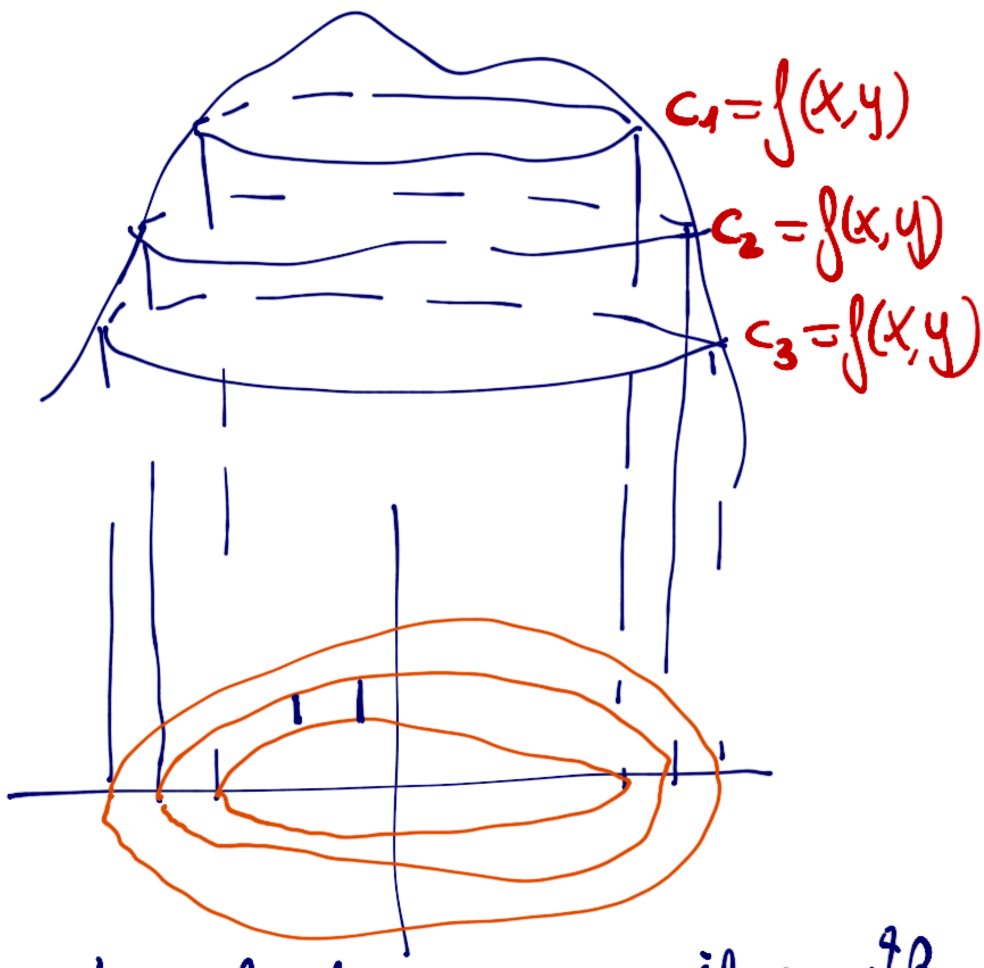
The level set of f at the value c as the set of points $x \in A$ such that

$$\boxed{f(x) = c}$$

- If $N=2$ $(\mathrm{Dom} f \subset \mathbb{R}^2)$ we find level curves
- If $N=3$ $(\mathrm{Dom} f \subset \mathbb{R}^3)$ we find level surfaces.

Remark

- The level curves are points of the domain where the function is constant.
- It allows us to draw 3D figures in 2D



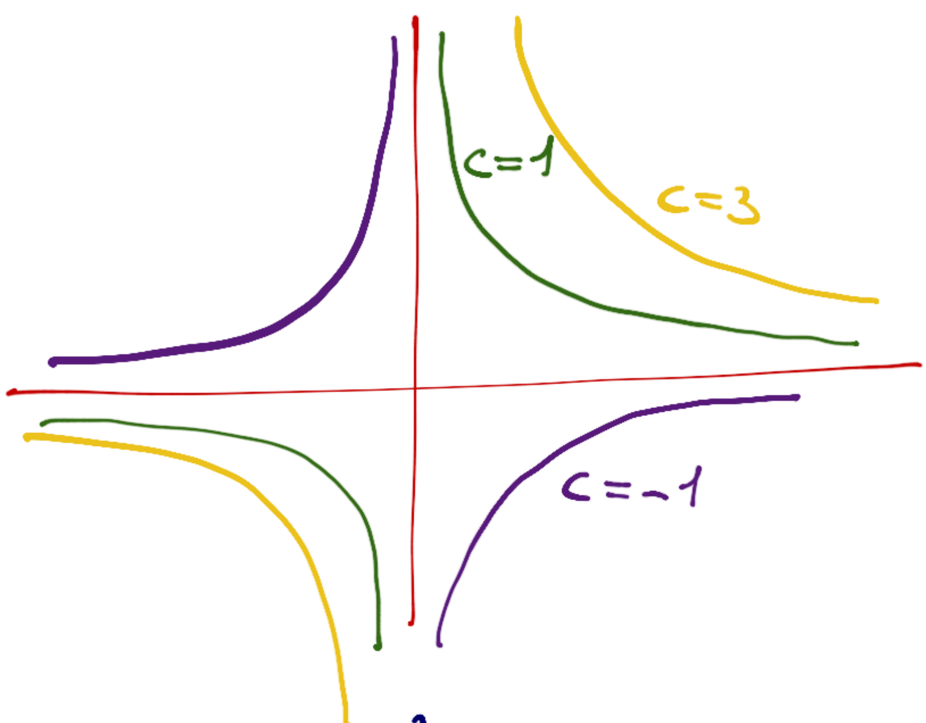
Distance between level curves provides with an idea of how quickly the function increases or decreases.

Problem 9 : $f(x,y) = xy$, $c = 1, -1, 3$

$c = 1 \Rightarrow xy = 1$; $y = \frac{1}{x}$

$c = -1 \Rightarrow xy = -1$; $y = -\frac{1}{x}$

$c = 3 \Rightarrow xy = 3$; $y = \frac{3}{x}$



Graph of a function

$\{ (x, f(x)) ; x \in \mathrm{Dom} f \}$

Example: Describe the graph of

$f(x,y) = x^2 + y^2$ $\quad f \equiv$ function

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}. \quad \mathrm{Dom} f = \mathbb{R}^2$

$f \geq 0$

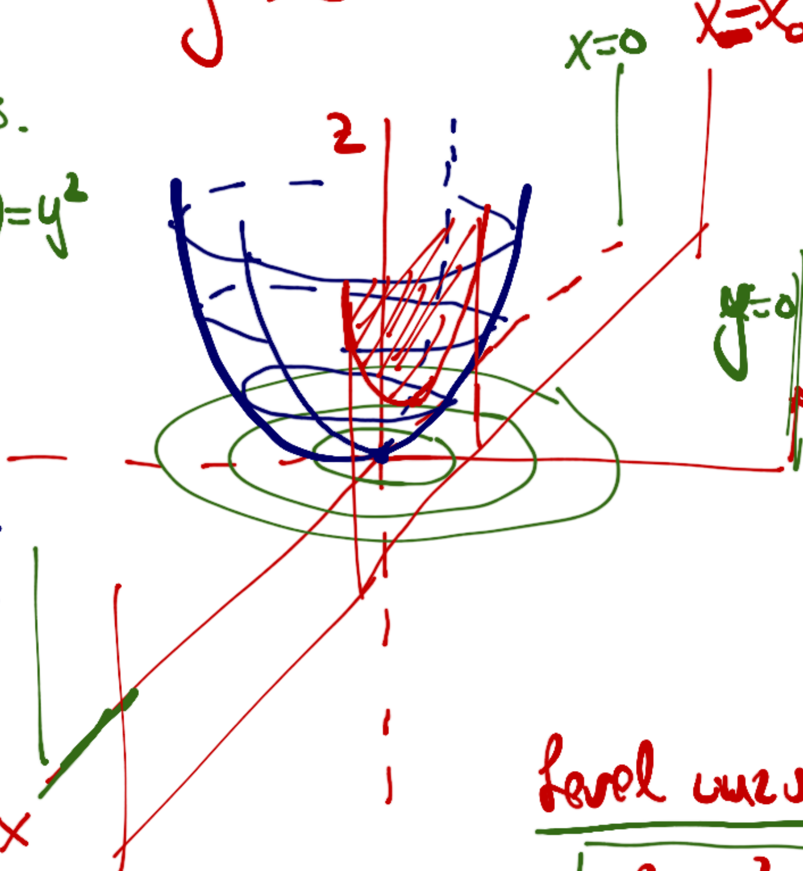
Vertical planes.

$\underline{x=0} \Rightarrow f(0,y) = y^2$

$\underline{x = x_0}$

$\Downarrow$

$f(x_0, y) = y^2 + \underline{x_0^2}$



$y=0, \; f(x,0) = x^2$

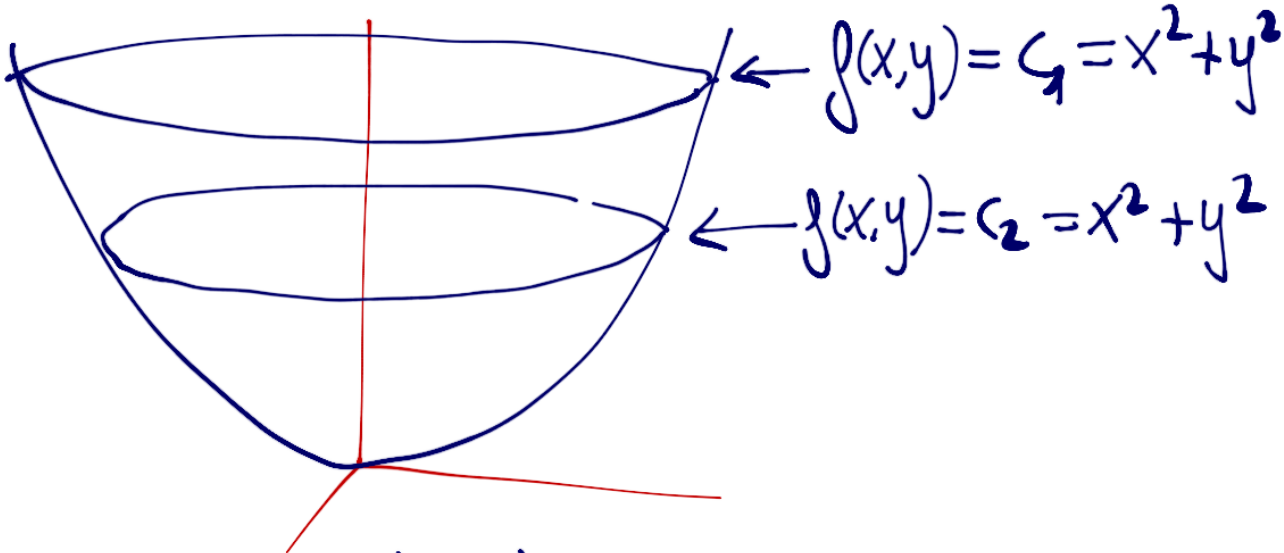
level curves:

$\boxed{x^2 + y^2 = C}$

$z = x^2 + y^2$ surface.

Paraboloid

circumferences of radius $\sqrt{c}$ and centered at the origin.



Limits and continuity

The limit of a function at a point x_0 will be the value that the function should be depending on the values around it.

$$\lim_{x \to x_0} f(x) = L$$

Definition $f: \mathbb{R}^N \to \mathbb{R}^M$, $x_0 \in \mathbb{R}^N$, $L \in \mathbb{R}^M$

We say that L is the limit of $f(x)$. when x is very close to x_0, $\lim_{x \to x_0} f(x) = L$, if for any $\varepsilon > 0$ there exists $\delta > 0$, $\|f(x) - L\| < \varepsilon$ when $\|x - x_0\| < \delta$.

How to compute the limits?

① Applying the definition.

For any $\varepsilon > 0$, $\exists\, \delta > 0$, $\delta \equiv \delta(\varepsilon)$ such that (for example in 2D) $f : \mathbb{R}^2 \to \mathbb{R}$.

$$0 < \|(x,y) - (x_0, y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

$$0 < (x-x_0)^2 + (y-y_0)^2 < \delta^2$$

$$\Downarrow$$

$$|f(x,y) - L| < \varepsilon \quad \longleftarrow$$

we start from here.

Example: $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\sin\frac{1}{x^2+y^2} = 0$

Find $\delta > 0$, we start from

$$|f(x,y) - 0| = |f(x,y)| = \left|\sqrt{x^2+y^2}\,\sin\frac{1}{x^2+y^2}\right|$$

$$\leq \sqrt{x^2+y^2} < \varepsilon = \delta$$

$$\delta = \delta(\varepsilon) = \varepsilon$$

and choosing those δ and ε, when (x,y) are suff. close to $(0,0)$ the function $f(x,y)$ will be suff. close to 0.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \boxed{0}$$

② Iterative limits

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = L$$

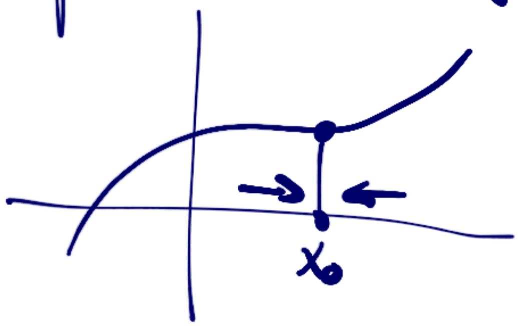
This method is valid to prove non-existence, but not to prove the value of the limit.

③ Directional limits. Approximation through a family of functions

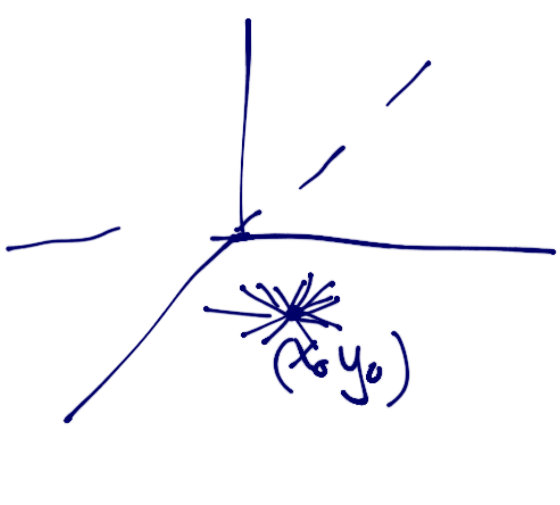
For example: $\left.\begin{array}{l} y=\alpha x \\ y=\alpha x^2 \\ \vdots \end{array}\right\}$

Remark

In one variable we used to compute one-sided limits to prove existence of a limit.



In several variables we have infinitely many directions so the approach following a family of functions does not justify the existence of the limit.



Example: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$

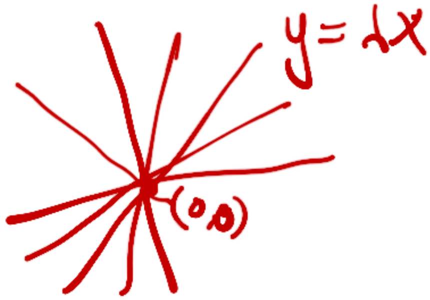
We choose the family of curves $y=\lambda x$,

$\lambda \equiv$ arbitrary parameter.

Then,

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+\lambda^2 x^2}} = \lim_{x\to 0} \frac{\pm 1}{\sqrt{1+\lambda^2}}$$

$$= \frac{x}{|x|\sqrt{1+\lambda^2}}$$



$$= \frac{\pm 1}{\sqrt{1+\lambda^2}}$$

The limit depends on λ, on the direction.

Therefore, the limit does not exist.

It must be unique.

Problem 16

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

i) $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = \lambda x$

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2} = \lim_{x\to 0} \frac{\lambda x^4}{x^6+\lambda^2 x^2}$$

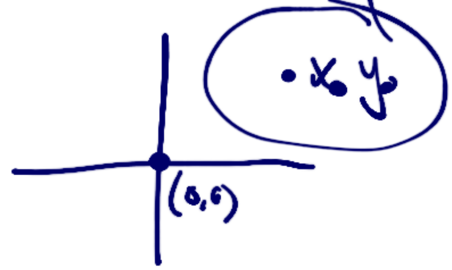
$$= \lim_{x\to 0} \frac{\lambda x^2}{x^4+\lambda^2} = 0 \Rightarrow$$ The limit following $y=\lambda x$ is 0 but it does not mean the limit exists.

ii) Compute the limit along $y = x^3$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^6}{x^6+x^6} = \frac{1}{2} \Rightarrow$$ The limit does not exist.

④ Limits using polar coordinates (in $\mathbb{R}^2$)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



Proposition

Let A be a subset in $\mathbb{R}^2$ such that $(0,0) \in A$ and $f: A \subset \mathbb{R}^2 \to \mathbb{R}$

If $\lim\limits_{r \to 0} f(r\cos\theta, r\sin\theta)$

depends on θ, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

For a general point $(x_0, y_0) \in A$ this is valid for the limit

$\lim\limits_{r\to 0} f(x_0 + r\cos\theta,\, y_0 + r\sin\theta)$ related to $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$

Proposition

$A \subset \mathbb{R}^N$, $x_0 \in A$, $f, g : A \subset \mathbb{R}^N \to \mathbb{R}$ two scalar functions.

If $\lim_{x \to x_0} f(x) = 0$ and g is bounded

$|g(x)| < C$, $x \in B(x_0, r)$

Then,

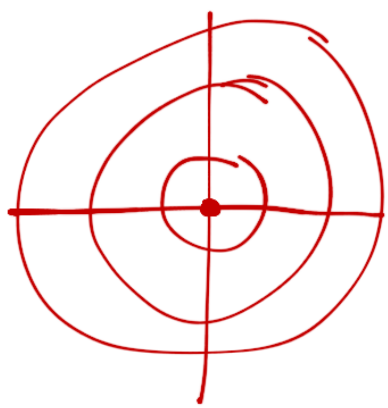
$$\lim_{x \to x_0} f(x) g(x) = 0$$

Example: $\underline{\lim_{(x,y) \to (0,0)}} \dfrac{xy^2}{x^2+y^2} =$

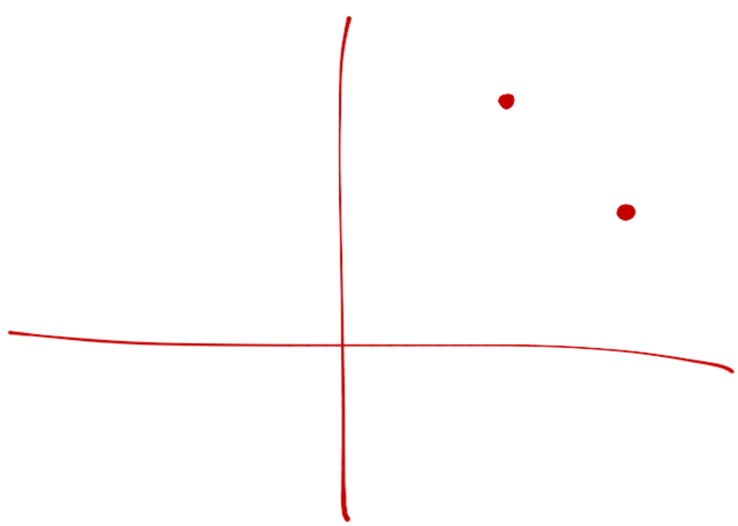
Assume $\boxed{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$

$$= \underline{\lim_{r \to 0}} \frac{r\cos\theta \, r^2 \sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r \to 0} \frac{r^3 \cos\theta \sin^2\theta}{r^2}$$

$$= \lim_{r \to 0} r \underbrace{\cos\theta \sin^2\theta}_{\text{bounded.}} = 0$$



Therefore we can conclude that the limit exists and it is actually 0.



Continuity

Definition

A subset of $\mathbb{R}^N$, $x_0 \in A$, $f : A \subset \mathbb{R}^N \to \mathbb{R}^M$

We say that f is continuous if

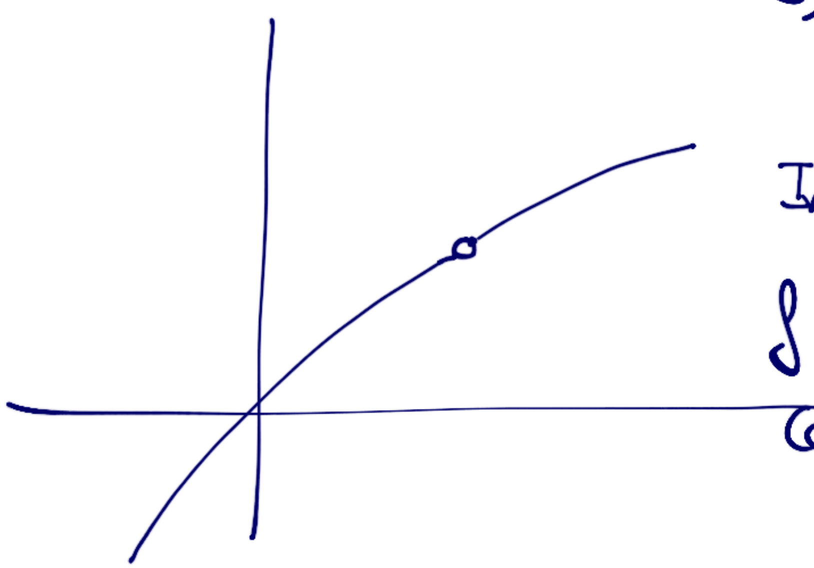
$$\lim_{x \to x_0} f(x) = f(x_0)$$

Assuming:

a) f is defined at x_0

b) The limit exists

c) $\lim_{x \to x_0} f(x) = f(x_0)$



In case $f(x) = (f_1(x), \ldots f_M(x))$

f will be cont. if every component is cont. at x_0

Example:

$$f(x,y) = x^2 \sin(x^2+y)$$

cont. since it is a composition, addition and multiplication of cont. functions.

$$F(x,y) = \left(x^2 y \sin(x^2+y), \frac{x}{x^2+y^2+1} \right)$$

Both components are cont. since $x^2+y^2+1 \neq 0$ for any $(x,y) \in \text{Dom } F$.